

GCSE / IGCSE Maths – Vectors and Proof

GCSE/IGCSE Higher · Total marks: 48 · Answers on the last page

Column vectors and vector geometry, algebraic proof and geometric proof at Higher tier. Non-calculator throughout; allow about 60 minutes and set out every proof in full.

Section A – Skills (Non-calculator.)

1. $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$.
 (a) Write $2\mathbf{a} + \mathbf{b}$ as a column vector.
 (b) Write $\mathbf{a} - 3\mathbf{b}$ as a column vector.
 (c) Find the magnitude of $\mathbf{a} - 3\mathbf{b}$, giving your answer in the form $k\sqrt{10}$. [5]
2. In triangle OAB , $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB .
 (a) Find $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
 (b) Show that $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. [3]
3. Prove that the sum of any three consecutive integers is a multiple of 3. [2]
4. Prove that $(n + 3)^2 - (n - 3)^2$ is a multiple of 12 for all integers n . [3]
5. Prove that the product of two odd numbers is always odd. [3]

Section B – Exam-style

6. In triangle OAB , $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Point C lies on OA with $OC : CA = 2 : 1$ and point D lies on AB with $AD : DB = 1 : 2$.
 (a) Find $\overrightarrow{CD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, simplifying your answer.
 (b) What does your answer tell you about the lines CD and OB ? [5]
7. AB is a diameter of a circle with centre O , and C is a point on the circumference. Prove that angle $ACB = 90^\circ$. You may use the fact that $OA = OB = OC$. [4]
8. Prove that $n^2 - n$ is even for every integer n . [3]
9. Prove that the difference between the squares of any two consecutive odd numbers is a multiple of 8. [4]

Section C – Challenge

10. $\overrightarrow{OA} = 6\mathbf{a}$ and $\overrightarrow{OB} = 6\mathbf{b}$. M is the point on OA with $OM : MA = 1 : 2$ and N is the point on AB with $AN : NB = 2 : 1$.
 (a) Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
 (b) Show that MN is parallel to OB and find the ratio $MN : OB$. [6]
11. P , Q and R have position vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} k \\ 11 \end{pmatrix}$. Find the value of k for which P , Q and R lie on a straight line. [4]

12. (a) A two-digit number has tens digit a and units digit b . Prove that the difference between the number and the number formed by reversing its digits is always a multiple of 9.
- (b) Prove that $x^2 - 6x + 10$ is positive for all real values of x . [6]

Answers

Check your working against these. If an answer does not match, try the question again before looking for the method.

- $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$
 - $\begin{pmatrix} 9 \\ -13 \end{pmatrix}$
 - $\sqrt{81 + 169} = \sqrt{250} = 5\sqrt{10}$
- $\mathbf{b} - \mathbf{a}$
 - $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
- $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$, which is a multiple of 3
- $(n^2 + 6n + 9) - (n^2 - 6n + 9) = 12n$, a multiple of 12
- $(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$, which is odd
- $\overrightarrow{CD} = \frac{1}{3}\mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}\mathbf{b}$
 - CD is parallel to OB and $CD = \frac{1}{3}OB$
- Triangles OAC and OBC are isosceles (radii). Let angle $OAC = x$ so angle $OCA = x$; let angle $OBC = y$ so angle $OCB = y$. Angles in triangle ABC : $2x + 2y = 180^\circ$, so $x + y = 90^\circ$, i.e. angle $ACB = 90^\circ$
- $n^2 - n = n(n - 1)$, the product of two consecutive integers; one of them is even, so the product is even
- $(2n + 3)^2 - (2n + 1)^2 = (4n^2 + 12n + 9) - (4n^2 + 4n + 1) = 8n + 8 = 8(n + 1)$
- $\overrightarrow{AB} = 6\mathbf{b} - 6\mathbf{a}$, $\overrightarrow{ON} = 6\mathbf{a} + \frac{2}{3}(6\mathbf{b} - 6\mathbf{a}) = 2\mathbf{a} + 4\mathbf{b}$
 - $\overrightarrow{MN} = (2\mathbf{a} + 4\mathbf{b}) - 2\mathbf{a} = 4\mathbf{b} = \frac{2}{3}\overrightarrow{OB}$, so parallel; $MN : OB = 2 : 3$
- $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} k - 2 \\ 10 \end{pmatrix}$; parallel when $\frac{k-2}{3} = \frac{10}{6}$, so $k = 7$
- $(10a + b) - (10b + a) = 9a - 9b = 9(a - b)$
 - $x^2 - 6x + 10 = (x - 3)^2 + 1 \geq 1 > 0$ since $(x - 3)^2 \geq 0$

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