IB Maths AI HL – Topic 1: Number and Algebra

IB AI HL · Total marks: 70 · Answers on the last page

Complex numbers in Cartesian, polar and Euler form, matrix algebra, eigenvalues and eigenvectors, and matrix transformations. A GDC is allowed throughout; allow about 70 minutes.

Section A – Core skills (Give exact answers unless otherwise stated.)

1. Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$.
   (a) Find $z_1 z_2$ in the form $a + bi$.
   (b) Find $\dfrac{z_1}{z_2}$ in the form $a + bi$.
   (c) Find $|z_1|$. [5]

2. Let $z = -1 + \sqrt{3}\,i$.
   (a) Write z in the form $r\,\mathrm{cis}\,\theta$ with $-\pi < \theta \leq \pi$.
   (b) Write z in Euler form $re^{i\theta}$.
   (c) Hence find z^6. [5]

3. Solve $z^3 = 8i$, giving the roots in the form $a + bi$ with exact values. [5]

4. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}$.
   (a) Find $\mathbf{AB}$.
   (b) Find $\det \mathbf{A}$.
   (c) Find $\mathbf{A}^{-1}$. [5]

5. Use the inverse matrix from question 4 to solve the simultaneous equations $2x + y = 7$ and $3x + 4y = 18$. Show the matrix method clearly. [3]

Section B – Exam-style

6. Let $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
   (a) Show that the characteristic equation of $\mathbf{M}$ is $\lambda^2 - 7\lambda + 10 = 0$.
   (b) Find the eigenvalues of $\mathbf{M}$.
   (c) Find a corresponding eigenvector for each eigenvalue. [7]

7. A triangle has vertices $P(1, 1)$, $Q(3, 1)$ and $R(1, 4)$. It is rotated $90°$ anticlockwise about the origin and the image is then reflected in the line $y = x$.
   (a) Write down the matrix for each transformation.
   (b) Find the single matrix representing the combined transformation, and describe it geometrically.
   (c) Find the coordinates of the image of the triangle. [7]

8. The transformation $\mathbf{T} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ is applied to a square of area 4 square units.
   (a) Find $\det \mathbf{T}$.
   (b) State the area of the image.
   (c) The image is a parallelogram. Explain why it cannot be a rectangle by considering the images of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. [5]

9. Using the eigenvalues and eigenvectors of $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ from question 6:
   (a) Write $\mathbf{M}$ in the form $\mathbf{PDP}^{-1}$, stating $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{P}^{-1}$.
   (b) Hence find an expression for $\mathbf{M}^n$.
   (c) Find $\mathbf{M}^5$. [8]

Section C – Extended

10. Each year 20 percent of the residents of town A move to town B, and 30 percent of the residents of town B move to town A. Initially town A has 1000 residents and town B has 0. Let $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ be the populations after n years.
    (a) Write down the transition matrix $\mathbf{T}$ such that $\mathbf{x}_{n+1} = \mathbf{T}\mathbf{x}_n$.
    (b) Find the populations after 1 year and after 2 years.
    (c) Find the eigenvalues of $\mathbf{T}$.
    (d) Hence find the long-term populations of the two towns. [8]

11. Two alternating voltages are $v_1 = 5\sin(\omega t)$ and $v_2 = 12\sin\left(\omega t + \frac{\pi}{2}\right)$, represented by the phasors (complex numbers) 5 and $12i$ respectively.
    (a) Write down the phasor representing $v_1 + v_2$.
    (b) Find the modulus and argument of this phasor, giving the argument in radians to 3 significant figures.
    (c) Hence write $v_1 + v_2$ in the form $R\sin(\omega t + \alpha)$. [6]

12. A transformation consists of a stretch with scale factor 3 parallel to the x-axis, followed by a rotation of $45°$ anticlockwise about the origin.
    (a) Find the single 2×2 matrix representing the combined transformation, giving entries in exact form.
    (b) Find the exact image of the point $(2, 0)$.
    (c) A shape of area A is transformed. Find the area of its image in terms of A. [6]

Answers

Check your working against these. If an answer does not match, try the question again before looking for the method.

1. (a) $11 - 2i$
   (b) $-1 + 2i$
   (c) 5

2. (a) $2\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$
   (b) $2e^{2\pi i/3}$
   (c) $z^6 = 64e^{4\pi i} = 64$

3. $8i = 8\,\mathrm{cis}\frac{\pi}{2}$; roots $2\,\mathrm{cis}\frac{\pi}{6}, 2\,\mathrm{cis}\frac{5\pi}{6}, 2\,\mathrm{cis}\frac{3\pi}{2}$, i.e. $z = \sqrt{3} + i,\ -\sqrt{3} + i,\ -2i$

4. (a) $\begin{pmatrix} 0 & 5 \\ -5 & 20 \end{pmatrix}$
   (b) 5
   (c) $\frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$

5. $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so $x = 2,\ y = 3$

6. (a) $\det(\mathbf{M} - \lambda\mathbf{I}) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10$
   (b) $\lambda = 5,\ \lambda = 2$
   (c) $\lambda = 5$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (any non-zero multiples)

7. (a) Rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, reflection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: a reflection in the x-axis
   (c) $P'(1, -1),\ Q'(3, -1),\ R'(1, -4)$

8. (a) 5
   (b) 20 square units
   (c) Images are $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; their scalar product is $3 + 2 = 5$
   $e0$, so the image sides are not perpendicular

9. (a) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$
   (b) $\mathbf{M}^n = \frac{1}{3}\begin{pmatrix} 2 \cdot 5^n + 2^n & 5^n - 2^n \\ 2 \cdot 5^n - 2^{n+1} & 5^n + 2^{n+1} \end{pmatrix}$
   (c) $\mathbf{M}^5 = \begin{pmatrix} 2094 & 1031 \\ 2062 & 1063 \end{pmatrix}$

10. (a) $\mathbf{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$
    (b) Year 1: $(800, 200)$; year 2: $(700, 300)$
    (c) $\lambda = 1$ and $\lambda = 0.5$
    (d) Steady state is the eigenvector for $\lambda = 1$, with $a : b = 3 : 2$, so 600 in A and 400 in B

11. (a) $5 + 12i$
    (b) Modulus 13, argument $\arctan\frac{12}{5} = 1.18$ rad (1.176)

(c) $v_1 + v_2 = 13\sin(\omega t + 1.18)$

12. (a) $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}3 & -1\\ 3 & 1\end{pmatrix}$
(b) $(3\sqrt{2}, 3\sqrt{2})$
(c) $\det = 3$, so area $3A$

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